**Lecture 3 exercises: Determinants and transformations**

1. Compute the determinants of the following matrices:

   $(a.)\ \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ $\quad (b.)\ \begin{bmatrix} -5 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad (c.)\ \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad (d.)\ \begin{bmatrix} -5 & 1 & -1 \\ 1 & 7 & -2 \\ 3 & 0 & 1 \end{bmatrix}$

2. Solve the following system of equations using determinants:

$$\begin{aligned} 4x + 5y - 2z &= 2 \\ 3x + 4y - z &= 6 \\ -x + 2y + 3z &= 1 \end{aligned}$$

3. Does a linear transformation always map the origin to the origin?

4. Find the $2 \times 2$ matrix that represents the linear transformation that first reflects a point in the line $x + y = 0$ and then rotates over $45^\circ$ about the origin.

5. Let $A$ be a $3 \times 3$ matrix that represents a linear transformation that maps the points $(2, 3, 2)$, $(1, 0, 2)$, and $(0, 2, 4)$ to the origin. What is $A$?

6. Let $A$ be a $2 \times 2$ matrix that represents a linear transformation where the mapping puts all points on the $x$-axis, while halving their $x$-coordinates. What is $A$?

7. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Try to find out what linear transformation $A$ represents, and describe it carefully in words.

8. Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Try to find out what linear transformation $A$ represents, and describe it carefully in words.

9. The shear transformation was introduced as the linear transformation represented by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. But in fact, $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ represents a shear transform for any value of $c$. Describe how $c$ influences the linear transformation.

10. Give the matrix in homogeneous coordinates of the affine transformation (in 2D) that represents rotation over $180^\circ$ over the point $(3, 1)$.

11. Give the matrix in homogeneous coordinates of the affine transformation (in 2D) that represents scaling by a factor 3 (for both coordinates) with respect to the point $(1, 1)$.

12. Give the matrix in homogeneous coordinates of the affine transformation (in 3D) that represents reflection in the plane $x + z = 3$.

13. Suppose that a linear transformation maps a point $(2, 3)$ to $(0, 1)$ and maps a point $(9, 7)$ to $(1, 0)$. Find the matrix for this linear transformation.

14. The absolute value of the determinant gives the area of a unit square after it is transformed by a linear transformation. Can a similar statement be made for *affine* transformations, where we take the matrix in homogeneous coordinates?

    If you can make this exercise without looking at the hint, you understand linear algebra very well!

    Hint: To answer this question, look carefully at the matrix of an affine transformations, and see what the value of its determinant is by using the cofactors of a suitable row.